Quantum illumination and quantum radar

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Introduction

- Quantum technologies has become a vast and highly active field of research: computation, cryptography, sensing, metrology...
- Making use of quantum features (superposition and entanglement) achieve otherwise impossible results, e.g., quadratic improvement in measurement sensitivities ($\propto \frac{1}{\sqrt{N}} \rightarrow \propto \frac{1}{N}$)
- Quantum sensing promising in near future applications using non-classical radiation fields and Gaussian state implementation, e.g., quantum illumination (QI)
- QI promises to outperform classical counterparts - even when
  - Low reflectivity
  - Low brightness
  - High thermal background
Classical radar

- Classical radar has not fundamentally changed in over 50 years. Obeys the *radar equation*:

\[
P_r = \frac{P_t G_t A_r \sigma F^4}{(4\pi)^2 R^4},
\]

- Introducing noise \( P_n = k_B T B_n F_n \), performance depends on \( \text{SNR} = P_r/P_n \)
- Vulnerable to a host of electronic countermeasures
### Basic QI protocol

1. **Entangle Signal and Idler mode pairs**
   - ![Diagram](source.png)

2. **Send Signal, retain Idler**
   - ![Diagram](source.png)

3. **Collect Return = Noise + Signal (?)**
   - ![Diagram](source.png)

4. **Measure Idler with Return**
   - ![Diagram](source.png)

5. **Detect:**
   - (i) **Correlations** $\Rightarrow$ Target Present
   - (ii) No correlations $\Rightarrow$ Target Absent
Quantum hypothesis testing

- Quantum radar is a task of binary QHT:
  - performance reduced to the distinguishability of states $\rho_\mu$.
- Requires $\rho_\mu \otimes M$ with $M >> 1$

\[
P_{err}^{QI} = \frac{1}{2} \exp \left( -\frac{M\kappa N_S}{N_B} \right) \tag{2}
\]
\[
P_{err}^{CS} = \frac{1}{2} \exp \left( -\frac{M\kappa N_S}{4N_B} \right) \tag{3}
\]
What we do...

- Generalise the definition of a quantum radar beyond QI:
  
  _Any model that exploits a quantum part or device to outperform a corresponding classical radar under the same conditions of energy, range, etc._

- We progressively relax entanglement requirements of QI and study the corresponding detection performances to the point where the source becomes just-separable, i.e., a maximally-correlated separable state.

- At the same time, try to formulate equivalences between figures of merit between classical and quantum radar performance.
We consider a source modelled as a two-mode Gaussian state:

\[ V_{\text{gen}}^{SI} = \begin{pmatrix} S & C \\ C & S \end{pmatrix} \oplus \begin{pmatrix} S & -C \\ -C & S \end{pmatrix}, \tag{4} \]

where

\[ S := N_S + 1/2, \]

\[ 0 \leq C \leq \sqrt{S^2 - 1/4} = \sqrt{N_S(N_S + 1)} = C_q, \]

\[ C_d = N_S. \]
The joint state of our returning \((R)\) mode and the retained idler is given by, under \(H_0\) and \(H_1\), respectively:

\[
\mathbf{V}_{RI}^{(0)} = \left( \begin{array}{cc} B & 0 \\ 0 & S \end{array} \right) \oplus \left( \begin{array}{cc} B & 0 \\ 0 & S \end{array} \right),
\]

\[
\mathbf{V}_{RI}^{(1)} = \left( \begin{array}{cc} A & \sqrt{\kappa}C \\ \sqrt{\kappa}C & S \end{array} \right) \oplus \left( \begin{array}{cc} A & -\sqrt{\kappa}C \\ -\sqrt{\kappa}C & S \end{array} \right),
\]

where \(B := N_B + 1/2\) and \(A := \kappa N_S + B\).
Coherent state scenario - no access to idler!

- Signal mode, annihilation operator $\hat{a}_S$, prepared in the coherent state $|\sqrt{N_S}\rangle$

- $H_0$: return is in a thermal state with mean photons per mode $N_B$, mean vector of zero and CM $(N_B + 1/2)\mathbf{1}_2$.

- $H_1$: return corresponds to a displaced thermal state with mean vector $(\sqrt{2\kappa N_S}, 0)$ and CM $(N_B + 1/2)\mathbf{1}_2$.

\[ H_0 : \hat{a}_R = \hat{a}_B, \quad N_B \gg 1 \]

\[ H_1 : \hat{a}_R = \sqrt{\kappa} \hat{a}_S + \sqrt{1 - \kappa} \hat{a}_B, \quad \kappa << 1, \quad N_B \to \frac{N_B}{1 - \kappa} \]
Hypothesis testing for quantum radar detection

• Detection probability $P_d := P(H_1|H_1)$.

• Two types of error may occur:
  type-I (false-alarm) $P_{fa} = P(H_1|H_0)$, and
  type-II (missed detection) $P_{md} = P(H_0|H_1)$.

• One may consider asymmetric testing in order to take in account discrepancies in error cost.

• A symmetric approach aims to obtain a global minimisation over all errors, irrespective of their origin. In this case, one considers the minimization of the average error probability

$$P_{err} := P(H_0)P(H_1|H_0) + P(H_1)P(H_0|H_1).$$ (7)
Symmetric hypothesis testing

- **Minimum** error probability is given by the Helstrom bound:
  \[ P_{\text{err}}^{\text{min}} = \left[ 1 - D(\hat{\rho}_0, \hat{\rho}_1) \right] / 2, \]
  where \( D(\hat{\rho}_0, \hat{\rho}_1) := |\hat{\rho}_0 - \hat{\rho}_1| / 2 \) is the trace distance.

- Analytically, use QCB,
  \[ P_{\text{err}}^{\text{min}} \leq P_{\text{err}}^{\text{QCB}} := \frac{1}{2} \left( \inf_{0 \leq s \leq 1} C_s \right), \quad C_s := (\hat{\rho}_0^s \hat{\rho}_1^{1-s}) . \tag{8} \]

- Forgoing minimization, set \( s = 1/2 \) and define a simpler, though weaker, upper bound, the QBB
  \[ P_{\text{err}}^{\text{QBB}} := \frac{1}{2} \left( \sqrt{\hat{\rho}_0} \sqrt{\hat{\rho}_1} \right) . \tag{9} \]

- For Gaussian states, closed analytical formulae exist for these!
Symmetric radar detection

- Assume typical condition of QI: $\kappa << 1$, $N_B >> 1$, $N_S << 1$.
- For a TMSV state, the minimum error probability satisfies

$$P_{\text{err}}^{\text{TMSV}} \leq e^{-M\kappa N_S/N_B}/2.$$  \hspace{1cm} (10)

- Computed using the QBB, exponentially tight in limit of large $M$.
- Error-rate exponent has a factor of 4 advantage over the corresponding coherent-state transmitter

$$P_{\text{err}}^{\text{CS}} \leq e^{-M\kappa N_S/4N_B}/2.$$  \hspace{1cm} (11)
Symmetric radar detection - generic source

- Extending Eq. (10) to the error probability for a generic source, we begin with $M = 1$ and assuming $\kappa \ll 1$, $N_B \gg 1$ and $N_S \ll 1$.

- The QBB takes the form

$$P_{\text{err}}^{\text{gen}} \leq e^{-\kappa N_S g_C(N_S)/N_B}/2, \quad (12)$$

where $g_C(N_S) \propto C^2$, the amount of correlations existing between the signal and idler modes.

- Demanding equivalence of exponents in the TMSV limit $C \rightarrow C_q$, we find that the QBB for $M$ probings becomes

$$P_{\text{err}}^{\text{gen}} \leq e^{-M \kappa N_S C^2/N_B C_q^2}/2. \quad (13)$$
Comparing Eqs. (13) and (11), we see that a quantum-correlated transmitter beats the coherent state transmitter if $P_{\text{err}}^{\text{gen}} \leq P_{\text{err}}^{\text{CS}}$ which means

$$C^2 \geq \frac{1}{4} \Rightarrow C \geq \frac{1}{2} \sqrt{N_S(N_S + 1)}. \quad (14)$$

Thus, the quadrature correlations required to outperform the semi-classical benchmark is half the value of those of a TMSV state.

At the separable limit $C = N_S$ - the relation is only satisfied for $N_S \geq 1/3$, contradicting the assumption $N_S \ll 1$.

A similar analysis holds if we relax the assumption of $N_S \ll 1$.

The employment of a source at the separable limit is not capable of beating coherent states under symmetric testing.
Asymmetric hypothesis testing

• Consider $M$ copies $\hat{\rho}_i^{\otimes M}$ of the state $\hat{\rho}_i$ encoding bit $i \in \{0, 1\}$.

• Binary outcome - two types of error, i.e., the type-I (false alarm) error

\[ P_{fa} := P(H_1|H_0) = (E_1\hat{\rho}_0^{\otimes M}) , \] (15)

and the type-II (missed detection) error

\[ P_{md} := P(H_0|H_1) = (E_0\hat{\rho}_1^{\otimes M}) . \] (16)

• These probabilities are dependent on the number $M$ of copies and, for $M \gg 1$, they both tend to zero, i.e.,

\[ P_{fa} \simeq e^{-\alpha_R M}, \quad P_{md} \simeq e^{-\beta_R M} , \] (17)

where we define the ‘error-exponents’ or ‘rate limits’ as

\[ \alpha_R = - \lim_{M \to +\infty} \frac{1}{M} \ln P_{fa}, \quad \beta_R = - \lim_{M \to +\infty} \frac{1}{M} \ln P_{md} . \] (18)
Asymmetric hypothesis testing 2

• Place a relatively loose constraint $P_{fa} < \epsilon$ on the type-I error, allowing us more freedom to minimize $P_{md}$.

• Quantum Stein’s lemma: given this constraint, QRE is the optimal decay rate for the type-II error probability

$$D(\hat{\rho}_0||\hat{\rho}_1) = [\hat{\rho}_0(\ln \hat{\rho}_0 - \ln \hat{\rho}_1)]$$

$$V(\hat{\rho}_0||\hat{\rho}_1) = [\hat{\rho}_0(\ln \hat{\rho}_0 - \ln \hat{\rho}_1)^2] - [D(\hat{\rho}_0||\hat{\rho}_1)]^2.$$  (19, 20)

• Tracking the type-II error exponent to second order (in $M$) depth, that is to order $\sqrt{M}$, allows one to define the QRE variance

• Optimal type-II error probability, for sample size $M$:

$$P_{md} = \exp \left\{ - \left[ MD(\hat{\rho}_0||\hat{\rho}_1) + \sqrt{MV(\hat{\rho}_0||\hat{\rho}_1)}\Phi^{-1}(\epsilon) + O(\log M) \right] \right\},$$

where $\epsilon \in (0, 1)$ bounds $P_{fa}$ and $\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y dx \exp \left( -x^2/2 \right).$
Asymmetric radar detection

- We evaluate the QRE and QRE-variance to first order in $N_B$ by taking an asymptotic expansion for large $N_B$ while keeping $N_S$ fixed. We obtain

$$D_{\text{gen}} := D \left( \hat{\rho}_{RI}^{(0)} || \hat{\rho}_{RI}^{(1)} \right) = \frac{\kappa C^2}{N_B} \ln \left( 1 + \frac{1}{N_S} \right) + \mathcal{O} \left( N_B^{-2} \right), \quad (22)$$

$$V_{\text{gen}} := V \left( \hat{\rho}_{RI}^{(0)} || \hat{\rho}_{RI}^{(1)} \right) = \frac{\kappa C^2(2N_S + 1)}{N_B} \ln^2 \left( 1 + \frac{1}{N_S} \right) + \mathcal{O} \left( N_B^{-2} \right). \quad (23)$$

- For coherent states these quantities take the form

$$D_{\text{CS}} := D \left( \hat{\rho}_{CS}^{(0)} || \hat{\rho}_{CS}^{(1)} \right) = \kappa N_S \ln \left( 1 + \frac{1}{N_B} \right) \approx \gamma + \mathcal{O} \left( N_B^{-2} \right), \quad (24)$$

$$V_{\text{CS}} := V \left( \hat{\rho}_{CS}^{(0)} || \hat{\rho}_{CS}^{(1)} \right) = \kappa N_S(2N_B + 1) \ln^2 \left( 1 + \frac{1}{N_B} \right) \approx 2\gamma + \mathcal{O} \left( N_B^{-2} \right), \quad (25)$$

- Signal-to-noise ratio (SNR): $\gamma := \frac{\kappa N_S}{N_B}$, usually expressed in decibels (dB) via $\gamma_{\text{dB}} = 10 \log_{10} \gamma$. 

**Asymmetric performance comparison**

Define an error-exponent advantage over coherent states:

\[
A(C, N_S) := \frac{D_{\text{gen}}}{D_{CS}} = \frac{C^2}{N_S \ln \left(1 + \frac{1}{N_S}\right)}.
\]  

(26)

- Benefits of max-entanglement for QI only for very small energies.
- For increasing \( N_S \), the ratio \( A \rightarrow 1 \), irrespective of source specification.
- Just-separable source quickly approaches the performance of QI already at about 20 photons.
Study mis-detection probability vs false-alarm probability for generic Gaussian source and coherent state classical benchmark.

- Optimal: phase is maintained - homodyne + coherent integration
- Deterministic phase shift - use heterodyne + coherent integration.
- Otherwise: unknown phase shift - heterodyne + non-coherent integration, given by Marcum’s Q-function
- Marcum’s Q-function may be overestimated by assuming single coherent pulse with $MN_S$ photons.
The ROC $P_{md} = P_{md}(P_{fa})$ of the generic quantum source can be upper bounded:

$$P_{md} \leq \tilde{P}_{md}^{gen} = \exp \left\{ - \left[ \sqrt{\frac{M\gamma}{N_S}} \Lambda \ln \left( 1 + \frac{1}{N_S} \right) + O(N_B^{-1}, 1) \right] \right\},$$

(27)

$$\Lambda := \left( \sqrt{\frac{M\gamma}{N_S}} C + \sqrt{2N_S + 1}\Phi^{-1}(P_{fa}) \right).$$

(28)

- Sufficiently large $M (\gtrsim 10^7)$ and large $N_B (\gtrsim 10^2)$. 

ROC: generic quantum source
ROC: coherent states

- Optimal - homodyne + coherent integration and binary testing:
  
  \[ P_{CS,\text{hom}}^{fa}(x) = \frac{1}{2} \text{erfc} \left( \frac{x}{\sqrt{M(2N_B + 1)}} \right), \]  
  \[ P_{CS,\text{hom}}^{md}(x) = \frac{1}{2} \text{erfc} \left( \frac{M\sqrt{2\kappa N_S} - x}{\sqrt{M(2N_B + 1)}} \right), \]  

  where \( \text{erfc}(z) := 1 - 2\pi^{-1/2} \int_0^z \exp(-t^2)dt \) is the complementary error function.

- Lower bound - assume single coherent state with mean number of photons \( MN_S \) so that the total SNR is given by \( M\gamma \):
  
  \[ P_{\text{Marcum}}^{md} = 1 - Q \left( \sqrt{2M\gamma}, \sqrt{-2\ln P_{fa}} \right), \]  

  where the Marcum Q-function is defined as
  
  \[ Q(x, y) := \int_y^\infty dt \, te^{-(t^2+x^2)/2} l_0(tx). \]
Focusing on short-range (\(\sim 1\)m) applications, e.g. security or biomedical:

- For \(\nu = 1\)GHz (L band) and \(T = 290\)K (room temperature), we get \(N_B \simeq 6 \times 10^3\) photons (bright noise).
- Assume broadband pulses, with 10\% bandwidth (100MHz), so that their individual duration \(\sim 10\)ns. Using \(M = 10^8\) pulses then we have an integration time \(\sim 1\)s - acceptable for slow-moving/still objects.
- Low-energy applications - assume \(N_S = 1\) mean photon per pulse.
- What about the SNR \(\gamma\)? - Related to overall reflectivity \(\kappa\), estimated by the radar equation.
The radar equation relates returning signal power $P_R$ to the transmitted signal power $P_T$:

$$P_R = \frac{GF^4 A_R \sigma}{(4\pi)^2 R^4} P_T$$  \hspace{1cm} (33)

At the same time, we can also write $\kappa$ as the ratio:

$$\kappa = \frac{P_R}{P_T} = \frac{GF^4 A_R \sigma}{(4\pi)^2 R^4},$$  \hspace{1cm} (34)

Assume $F = 1$ (no free-space loss) and ideal pencil beam so that solid angle $\delta$ is exactly subtended by the target’s $\sigma$ (valid at short range) so that $G = 4\pi / \delta = 4\pi R^2 / \sigma$.

Then,

$$\kappa = \frac{A_R}{(4\pi R)^2}, \quad R = \frac{1}{4\pi} \sqrt{\frac{A_R}{\kappa}}.$$  \hspace{1cm} (35)
ROC comparison: \( N_S = 1, R = 1m \)

- Red curves: Gaussian state QI with \( C(p) = pC_d + (1 - p)C_q \).
  - Just-separable, \( p = 0 \), (dotted), maximal entanglement \( (p = 1) \), solid, and
  - intermediate correlations \( (p = 1/6) \), dashed.
- Black curves: Classical coherent state benchmark
  - Optimal homodyne detection, thick, and lower bound, thin.
ROC comparison: \( N_S = 0.01, R = 0.1 \, m \)

- Red curves: Gaussian state QI with \( C(p) = pC_d + (1 - p)C_q \). Just-separable, \( p = 0 \), (dotted), maximal entanglement (\( p = 1 \)), solid, and intermediate correlations (\( p = 1/2 \)), dashed.

- Black curves: Classical coherent state benchmark Optimal homodyne detection, thick, and lower bound, thin.
Concluding remarks

- We have investigated how to loosen QI transmitter requirements.
- Scenarios of symmetric and asymmetric testing where we test the quantum performance with respect to suitable classical benchmarks.
- Quantum advantage still exists by using Gaussian sources which are not necessarily maximally entangled.
- Short ranges only!
  (so spherical beam spreading does not involve too many dBs of loss, a major killing factor for any quantum radar design based on the exploitation of quantum correlations)
- A short-range, low-power radar is potentially interesting as noninvasive scanning tool for biomedical applications but also for security and safety purposes.
Current and ongoing work

- Previous results assumes use of *optimal* receivers achieving

$$R_{QI} = \frac{\kappa N_S}{N_B} \quad \text{vs.} \quad R_{CS} = \frac{\kappa N_S}{4N_B}$$

(36)

- Practical receiver designs include Guha Erkman (2009) - Optical Parametric Amplifier (OPA) and Phase Conjugating (PC) receiver achieving

$$R_{PC/OPA} \simeq \frac{\kappa N_S}{2N_B}, \quad N_S \ll 1, \kappa \ll 1, N_B \gg 1$$

(37)

and Zhuang (2016) FF-SFG receiver (non-linear)
Source:

\[ \mathbf{V}_{S,l} = \frac{1}{2} \begin{pmatrix} \nu \mathbf{1} & c \mathbf{Z} \\ c \mathbf{Z} & \mu \mathbf{1} \end{pmatrix}, \quad \begin{cases} \mathbf{1} := \text{diag}(1, 1), \\ \mathbf{Z} := \text{diag}(1, -1), \end{cases} \tag{38} \]

where \( \nu := 2N_S + 1, \mu := 2N_I + 1 \) and \( c \) quantifies the quadrature correlations between the two modes such that \( 0 \leq c \leq 2\sqrt{N_S(N_I + 1)} \).

Return:

\[ \mathbf{V}^0_{R,l} = \frac{1}{2} \begin{pmatrix} \omega \mathbf{1} & 0 \\ 0 & \mu \mathbf{1} \end{pmatrix}, \tag{39} \]

\[ \mathbf{V}^1_{R,l} = \frac{1}{2} \begin{pmatrix} \gamma \mathbf{1} & \sqrt{\kappa c} \mathbf{Z} \\ \sqrt{\kappa c} \mathbf{Z} & \mu \mathbf{1} \end{pmatrix}, \tag{40} \]

where we set \( \omega := 2N_B + 1 \) and \( \gamma := 2\kappa N_S + \omega \).
SNR for generic source, $c$, is given by

$$\text{SNR}_{PC} = \frac{\kappa c^2}{\left( \sqrt{\kappa c^2 + \mu(1+\gamma)} + \sqrt{\mu(1+\omega)} \right)^2}. \quad (41)$$

This directly relates to its error probability after $M$ uses, for equally-likely hypotheses, satisfying

$$P_{PC}^{(M)} = \frac{1}{2} \text{erfc} \left( \sqrt{M\text{SNR}_{PC}} \right). \quad (42)$$
Using our SNR, write

\[ \mu \rightarrow \mu' = \mu + \varepsilon_I, \]  

(43)

and

\[ \omega \rightarrow \omega' = \omega + \varepsilon_R, \text{ under } H_0, \]
\[ \gamma \rightarrow \gamma' = \gamma + \varepsilon_R, \text{ under } H_1. \]  

(44)

Assume \( \varepsilon_{I(R)} = 1 \) (heterodyne).

QI+PC: Entangled TMSV source with PC receiver
QI+Het+PC: \( \varepsilon_I = \varepsilon_R = 1 \) before the PC receiver
QI+Cal+PC: \( \varepsilon_R = 1 \) and \( \varepsilon_I = 0 \)
Noisy receiver performance

Assume $\varepsilon_{I(R)} = 1$ (heterodyne).

QI+PC: Entangled TMSV source with PC receiver

QI+Het+PC: $\varepsilon_I = \varepsilon_R = 1$ before the PC receiver

QI+Cal+PC: $\varepsilon_R = 1$ and $\varepsilon_I = 0$

$$\text{SNR}_{\text{QI}+\text{Cal}+\text{PC}} \rightarrow \text{SNR}_{\text{QI}+\text{PC}} = \frac{(1 + N_I)\kappa N_S}{2N_B(1 + 2N_I)}, \quad \text{(45)}$$

and

$$\text{SNR}_{\text{QI}+\text{Het}+\text{PC}} \rightarrow \text{SNR}_{\text{CS}+\text{Hom}} = \frac{\kappa N_S}{4N_B}. \quad \text{(46)}$$

The maximal advantage of QI+PC over CS+Hom is given by

$$\frac{\text{SNR}_{\text{QI}+\text{PC}}}{\text{SNR}_{\text{CS}+\text{Hom}}} = \frac{2(1 + N_I)}{1 + 2N_I} \rightarrow 2 \text{ for } N_I \ll 1. \quad \text{(47)}$$
Thanks for listening!